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Non-equilibrium critical relaxation with coupling to a conserved density

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Abstract. We study non-equilibrium critical relaxation properties of model C (purely dissipative relaxation of an order parameter coupled to a conserved density) starting from a macroscopically prepared initial state with short-range correlations. Using a field-theoretic renormalization group approach we show that all the stages of growth of the correlation length display universal behaviour governed by a new critical exponent θ. This exponent is calculated to second order in $\epsilon = 4 - d$ where d is the spatial dimension of the system.

1. Introduction

The equilibrium state of a thermodynamic system whose parameters approach a critical point is characterized by long-range correlations. An interesting topic in the theory of dynamic critical phenomena is the relaxation of such a system from a non-equilibrium initial state with a *small* correlation length. This initial state can be prepared (at least in a computer experiment [1, 2]) by quenching the system from a high temperature $T_0 \gg T_c$ to T_c.

It is well known that within the critical region a variety of quantities such as correlation and response functions show universal scaling behaviour. An important goal of the theory is, therefore, the determination of the scaling laws that govern the non-equilibrium critical relaxation. A few years ago Janssen, Schaub and Schmittmann (JSS) [3] (see also [4]) showed that universality is not restricted to late times after the quench [5, 6] and that even the (macroscopically) early stages of the relaxation process display a universal 'initial slip' scaling behaviour. They obtained the following results in the case of purely dissipative relaxation of an order parameter (model A in the terminology of Halperin *et al* [7]):

The correlation function reads

$$C(r; t, t') = r^{-(d-2+\eta)} \left(\frac{t}{t'}\right)^{\theta-1} F_C\left(\frac{r}{\xi}, \frac{t}{\xi^z}\right) \qquad \text{for } t' \to 0 \tag{1}$$

where ξ denotes the equilibrium correlation length. The linear response of the order parameter to an external field is given by

$$\chi(r; t, t') = r^{-(d-2+\eta+z)} \left(\frac{t}{t'}\right)^{\theta} F_\chi\left(\frac{r}{\xi}, \frac{t}{\xi^z}\right) \qquad \text{for } t' \to 0. \tag{2}$$

If we allow for non-zero initial magnetization M_0 the initial growth and the decay of the order parameter display scaling behaviour

$$M(t) = M_0 t^{\theta'} f_M(t^{\theta'+\beta/(\nu z)} M_0) \tag{3}$$

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(at the critical point) with an exponent

$$\theta' = \theta + (2 - z - \eta)/z.$$

In the limit of very large or very small arguments the scaling function f_M is given by

$$f_M(x) \sim \begin{cases} 1 & \text{for } x \to 0 \\ 1/x & \text{for } x \to \infty. \end{cases}$$

Meanwhile f_M has been calculated and the corrections to the equilibrium value of $C(r; t, t)$ has been determined by one of us to first order in $\epsilon = 4 - d$ [4] (d is the dimension of physical space). In the case of model A the exponents θ and θ' are universal and new, i.e. they cannot be expressed in terms of exponents known from statics or equilibrium dynamics.

The 2-loop calculations performed by JSS are in good agreement with Monte Carlo experiments [1, 2] which 'measure' the autocorrelation function $C(t) = \langle s(r, t)s(r, 0)\rangle$ at the critical point. The exponent θ' can be obtained from the decay of the autocorrelation function according to the power law

$$C(t) \sim t^{-(d/z)+\theta'}. \tag{4}$$

Recently the order-parameter relaxation in a system of finite size [8] and the effect of quenched random impurities on the growth of correlations [9] have been studied. In this work we consider the effect of a non-critical conserved density coupling to the order parameter. Such conserved densities are abundant: energy, annealed random impurities, particle densities in binary mixtures, etc. It is very well known that the presence of a conserved density modifies the critical dynamics if the specific heat exponent $\alpha > 0$. Here we will find the following results:

The short time scaling forms (1) and (2) remain valid and the relaxation of a non-conserved order parameter is still given by equation (3), but the exponents θ and θ' depend on the dynamic universality class of the system. As in the case of model A, the initial slip exponent is universal and new.

We use dynamic field theory [10–12] to calculate θ for model C [7, 13] to second order in $\epsilon = 4 - d$.

2. The model

The dynamics of an order-parameter field s coupled to a non-critical conserved density m can be expressed in the form of the Langevin equations

$$\begin{aligned} \partial_t s(r, t) &= -\lambda \frac{\delta \mathcal{H}}{\delta s(r, t)} + \zeta(r, t) \\ \partial_t m(r, t) &= \lambda \rho \Delta \frac{\delta \mathcal{H}}{\delta m(r, t)} + \eta(r, t) \end{aligned} \tag{5}$$

with the Hamiltonian

$$\mathcal{H}[s, m] = \int \mathrm{d}^d r \left[\frac{\tau}{2} s^2 + \frac{1}{2} (\nabla s)^2 + \frac{g}{4!} s^4 + \frac{1}{2} m^2 + \frac{\gamma}{2} m s^2 - h_m m \right]. \tag{6}$$

We adjust the external field h_m in such a way that $\langle m\rangle = 0$. ζ and η are Gaussian random forces with zero mean and the correlations

$$\begin{aligned}\langle\zeta(r,t)\zeta(r',t')\rangle &= 2\lambda\delta(r-r')\delta(t-t')\\ \langle\eta(r,t)\eta(r',t')\rangle &= -2\lambda\rho\Delta\delta(r-r')\delta(t-t').\end{aligned} \tag{7}$$

Since we are interested in the relaxation of the system from a non-equilibrium initial state we have to specify the distribution of the fields $s_0(r) = s(r, t = 0)$ and $m_0(r) = m(r, t = 0)$. A quench from a temperature $T_0 \gg T_c$ corresponds to

$$\mathcal{P}[s_0, m_0] \propto \exp(-\mathcal{H}^{(i)}[s_0, m_0]) \tag{8}$$

where

$$\mathcal{H}^{(i)}[s_0, m_0] = \int \mathrm{d}^d r \left[\frac{\tau_0}{2}s_0^2 + \frac{1}{2c_0}m_0^2\right]. \tag{9}$$

This guarantees that the initial correlations are short range:

$$\begin{aligned}\langle s_0(r)s_0(r')\rangle &= \tau_0^{-1}\delta(r-r')\\ \langle m_0(r)m_0(r')\rangle &= c_0\delta(r-r').\end{aligned} \tag{10}$$

Further terms in $\mathcal{H}[s_0, m_0]$ turn out to be irrelevant.

An equivalent formulation for the dynamics is given by the stochastic functional [14]

$$\begin{aligned}\mathcal{J}[\tilde{s}, s; \tilde{m}, m] = \int_0^\infty \mathrm{d}t \int \mathrm{d}^d r \bigg\{&\tilde{s}\left[\partial_t s + \lambda(\tau-\Delta)s + \frac{\lambda g}{3!}s^3 + \lambda\gamma s m\right]\\ &- \lambda\tilde{s}^2 + \tilde{m}\partial_t m - \lambda\rho(\Delta\tilde{m})\left(\frac{\gamma}{2}s^2 + m\right) - \lambda\rho(\nabla\tilde{m})^2\bigg\}\end{aligned} \tag{11}$$

where $\tilde{s}$ and $\tilde{m}$ are the Martin–Siggia–Rose response fields [15].

The weight $\exp(-\mathcal{J})$ integrated over the response fields may be interpreted as the probability of a realization of the stochastic process $\{s(r,t), m(r,t)\}$ starting from an initial configuration $\{s_0(r), m_0(r)\}$ of the order parameter and the conserved density. In order to calculate correlation and response functions we have to average over the random forces and the initial configurations. For this purpose functional integrations of the form

$$\langle s(r,t)\ldots\rangle = \int \mathcal{D}[\mathrm{i}\tilde{s}, s; \mathrm{i}\tilde{m}, m] s(r,t)\ldots\exp(-\mathcal{J}[\tilde{s}, s; \tilde{m}, m] - \mathcal{H}^{(i)}[s_0, m_0])$$

have to be performed. The brackets $\langle\ldots\rangle$ indicate an average with respect to thermal noise and initial conditions.

As in the case of model A [3] the fixed point of the 'initial temperature' τ_0 under renormalization group transformations can be found by dimensional analysis. Since $\tau_0 \sim \mu^2$ (where μ^{-1} is an external length scale) the fixed points are $\tau_0^* = 0$ and $\tau_0^* = \pm\infty$, but only $\tau_0^* = +\infty$ corresponds to a normalizable probability distribution.

These simple arguments do not hold for the (dimensionless) constant c_0, so further work is necessary to obtain its fixed point. We defer this to the next section.

3. Renormalization

As a first step in the study of the non-equilibrium relaxation of model C we have to calculate the correlators and propagators of the Gaussian theory. The Gaussian model serves as the 'free part' of a perturbation series which we will analyse by the methods of renormalized field theory [10, 12].

For the Gaussian part one easily finds

$$\begin{aligned} G_q(t,t') :&= \int \mathrm{d}^d r\, \mathrm{e}^{-\mathrm{i}qr} \langle s(r,t)\tilde{s}(\mathbf{0},t')\rangle \\ &= \exp(-\lambda(\tau+q^2)(t-t')) \qquad \text{for } t>t' \end{aligned} \tag{12}$$

$$\begin{aligned} C_q(t,t') :&= \int \mathrm{d}^d r\, \mathrm{e}^{-\mathrm{i}qr} \langle s(r,t)s(\mathbf{0},t')\rangle \\ &= C_q^{(\mathrm{eq})}(t-t') + C_q^{(\mathrm{i})}(t,t') \end{aligned} \tag{13}$$

with the equilibrium correlator

$$C_q^{(\mathrm{eq})}(t-t') = \frac{1}{\tau+q^2}\exp(-\lambda(\tau+q^2)|t-t'|) \tag{14}$$

and a non-equilibrium ('initial') part

$$C_q^{(\mathrm{i})}(t,t') = \left(\tau_0^{-1} - \frac{1}{\tau+q^2}\right)\exp(-\lambda(\tau+q^2)(t+t')). \tag{15}$$

Thus the fixed point $\tau_0^{-1} = 0$ corresponds to sharp Dirichlet initial conditions $s_0 = 0$ for the order-parameter field. Analogously,

$$\begin{aligned} G_{m,q}(t-t') :&= \int \mathrm{d}^d r\, \mathrm{e}^{-\mathrm{i}qr} \langle m(r,t)\tilde{m}(\mathbf{0},t')\rangle \\ &= \exp(-\lambda\rho q^2(t-t')) \qquad \text{for } t>t' \end{aligned} \tag{16}$$

$$\begin{aligned} C_{m,q}(t,t') :&= \int \mathrm{d}^d r\, \mathrm{e}^{-\mathrm{i}qr} \langle m(r,t)m(\mathbf{0},t')\rangle \\ &= C_{m,q}^{(\mathrm{eq})}(t-t') + C_{m,q}^{(\mathrm{i})}(t,t') \end{aligned} \tag{17}$$

with

$$C_{m,q}^{(\mathrm{eq})}(t-t') = \exp(-\lambda\rho q^2|t-t'|) \tag{18}$$

$$C_{m,q}^{(\mathrm{i})}(t,t') = (c_0-1)\exp(-\lambda\rho q^2(t+t')). \tag{19}$$

More complicated correlation functions can be calculated by means of Wick's theorem. From now on we set $\tau_0^{-1} = 0$.

Since the use of the field-theoretic formalism in the theory of critical phenomena is described in several text books [16] we restrict ourselves to the main steps of the calculation.

First, ultraviolet divergent integrals which occur in the perturbation series have to be rendered finite by a regularization procedure. We choose dimensional regularization, i.e. the integrations are performed for dimensions $d = 4 - \epsilon$ where no divergence is present. The results are analytically continued to other dimensions. Renormalizability of the theory guarantees that the remaining poles at $\epsilon = 0$ can be absorbed into a finite number of reparametrizations of coupling constants and fields. In the theory of equilibrium critical dynamics these renormalizations are [17, 18]

$$\begin{aligned} &\mathring{s} = Z_s^{1/2} s \qquad \mathring{\tilde{s}} = Z_{\tilde{s}}^{1/2} \tilde{s} \\ &\mathring{m} = Z_m^{1/2} m \qquad \mathring{\tilde{m}} = Z_m^{-1/2} \tilde{m} \\ &Z_s \mathring{\tau} = Z_\tau \tau \qquad G_\epsilon Z_s^2 Z_m \mathring{\gamma}^2 = Z_\gamma^2 \gamma^2 \mu^\epsilon \\ &G_\epsilon Z_s^2 \mathring{g} = (Z_u u + 3 Z_\gamma^2 / Z_m \gamma^2) \mu^\epsilon \qquad U = u + 3\gamma^2 \\ &\mathring{\lambda} = (Z_s / Z_{\tilde{s}})^{1/2} Z_\lambda \lambda \qquad \mathring{\lambda} \mathring{\rho} = Z_\lambda Z_\rho Z_m \lambda \rho. \end{aligned} \tag{20}$$

Bare quantities are indicated by '∘'; the geometrical factor $G_\epsilon = \Gamma(1 + \epsilon/2)/(4\pi)^{d/2}$ has been introduced for convenience.

As a result of the simple quadratic structure of the Hamiltonian with respect to the conserved density there are several relations between (20) and the renormalization of model A. Z_s and Z_u depend on u only and are the same as in model A. Moreover $Z_\gamma = Z_\tau$, where Z_τ differs from the temperature renormalization in model A by a factor Z_m [18]. The identity $Z_\lambda = Z_\rho = 1$ follows from a dissipation fluctuation theorem [12].

The Z factors defined above suffice to absorb the ϵ poles in a field theory which is translationally invariant with respect to both space *and* time. However, the imposed *non-equilibrium initial conditions break the translational invariance with respect to time.* It is well known from the theory of surface critical phenomena [19, 20] that boundary conditions require additional renormalizations of surface coupling constants and fields. Analogously we have to subtract ultraviolet divergences which are located in the 'time surface' $t = 0$. These divergences appear in the non-translationally invariant contributions to the perturbation series which result from the initial parts (15), (19) of the correlators and from the restriction of internal time integrations to positive times $t \geqslant 0$. Compared with translationally invariant integrals such contributions contain one more time integration which is effective in decreasing the degree of divergence. By naive power counting time scales as $t \sim (\lambda\mu^2)^{-1}$ for a non-conserved order parameter. We therefore conclude that the absence of translational invariance with respect to time decreases the superficial degree of divergence by two.

As in the case of a semi-infinite system the renormalization of the (one-particle irreducible) vertex functions alone does not suffice to produce a well defined renormalized theory. Therefore we have to study the full one-particle reducible Green functions

$$G_{\tilde{N},N;\tilde{M},M} := \langle [s]^N [\tilde{s}]^{\tilde{N}} [m]^M [\tilde{m}]^{\tilde{M}} \rangle. \tag{21}$$

The above considerations show that we have to expect new (logarithmic) divergences in the two-point Green functions $\langle \tilde{s} s \rangle$ and $\langle ss \rangle$ since they are quadratically divergent in a translationally invariant theory. These divergences are suppressed in graphs where a vertex is connected to an external point by a correlator line because the correlator tends to zero as one of its time arguments approaches the 'surface' $t = 0$. Thus we only need to renormalize the response function $\langle s(r, t) \tilde{s}(r', 0) \rangle$ where one external point is fixed to $t' = 0$. The two-point correlation function $\langle s(r, t) s(r', 0) \rangle$ vanishes identically as a result of the Dirichlet

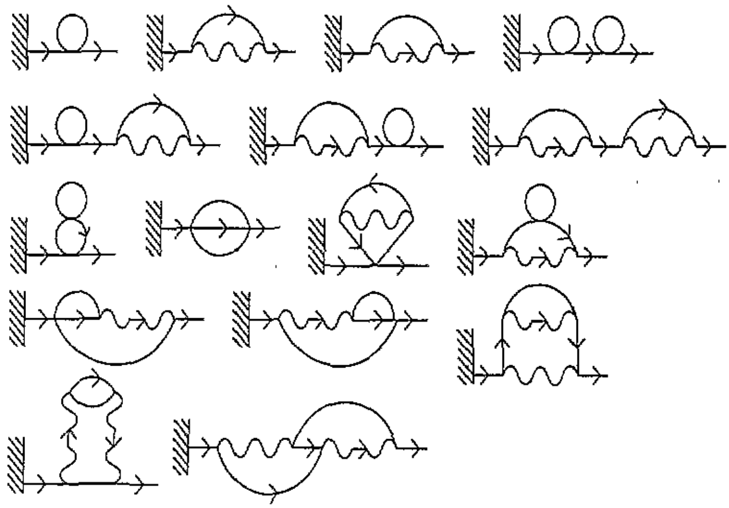

Figure 1. Typical Feynman graphs contributing to $\Gamma^{(i)}_{1,0}(q,t)_{[\tilde{s}_0]}$ at 2-loop order. The hatched area corresponds to the 'time surface' $t = 0$.

initial conditions. Insertions of the time derivative $\dot{s}_0(r) = \partial_t s(r,t)|_{t=0}$ in Green functions are related to the response field $\tilde{s}_0(r) = \tilde{s}(r, t = 0)$ by

$$\dot{s}_0(r) = 2\lambda\tilde{s}_0(r). \tag{22}$$

This has been verified by JSS who considered the corresponding Feynman graphs. Away from the fixed point $\tau_0^* = \infty$ insertions of the field $s_0(r)$ are different from zero and we have

$$s_0(r) = \tau_0^{-1}\tilde{s}_0(r). \tag{23}$$

For the conserved density one finds at $t = 0$

$$m_0(r) = c_0\tilde{m}_0 \tag{24}$$

and, for $c_0 = 0$,

$$\dot{m}_0(r) = -2\lambda\rho\Delta\tilde{m}_0(r). \tag{25}$$

Consequently there is only one new Z factor, Z_0, required to renormalize Green functions with external legs fixed to $t = 0$:

$$\tilde{s}_0 \to \mathring{\tilde{s}}_0 = (Z_0 Z_{\tilde{s}})^{1/2}\tilde{s}_0. \tag{26}$$

For the determination of Z_0 it is convenient to write the response function in the following form [3]:

$$\begin{aligned} G^{\mathrm{i}}_{0,1}(q,t) &:= \int \mathrm{d}^d r\, \mathrm{e}^{-\mathrm{i}qr}\langle s(r,t)\tilde{s}(0,0)\rangle \\ &= \int_0^t \mathrm{d}t'\, \bar{G}_{1,1}(q;t,t')\Gamma^{(i)}_{1,0}(q,t')_{[\tilde{s}_0]} \end{aligned} \tag{27}$$

where $\bar{G}_{1,1}(q;t,t')$ is the full unrenormalized response function calculated only with the equilibrium parts (14) and (18) of the correlators. $\Gamma^{(i)}_{1,0}(q,t)_{[\tilde{s}_0]}$ is a reducible 1-point vertex function with an amputated $\tilde{s}$-leg and a single insertion of the response field $\tilde{s}_0$. It contains in its last irreducible part at least one initial correlator (15) or (19). Figure 1 shows some of the Feynman graphs contributing to $\Gamma^{(i)}_{1,0}(q,t)_{[\tilde{s}_0]}$ at 2-loop order.

Although $\bar{G}_{1,1}(q;t,t')$ is calculated with equilibrium propagators and correlators it is different from the translational invariant equilibrium response function $G_{1,1}^{(\mathrm{eq})}(q;t-t')$. This is due to the restriction of internal time integrations to positive times $t \geqslant 0$.

Yet we can relate $G_{1,1}^{(\mathrm{eq})}(q;t-t')$ to $\bar{G}_{1,1}(q;t,t')$ by an equation similar to (27). Usually one calculates equilibrium Green functions by means of a stochastic functional like (11) where the lower limit of time integration ($t = 0$) is replaced by $t = -\infty$. Ergodicity of the dynamics then implies that the initial conditions at $t = -\infty$ do not affect equilibrium averages. We obtain the same results if we choose the equilibrium distribution $\mathcal{P}_{\mathrm{eq}}[s_0, m_0] \propto \exp(-\mathcal{H}[s_0, m_0])$ instead of (8) to average over the initial fields s_0 and m_0. In this way one gets a perturbation series for $G_{1,1}^{(\mathrm{eq})}(q;t-t')$ where internal time integrations range from zero to infinity. Nevertheless $G_{1,1}^{(\mathrm{eq})}(q;t-t')$ differs from $\bar{G}_{1,1}(q;t,t')$ because the non-Gaussian probability distribution $\mathcal{P}_{\mathrm{eq}}[s_0, m_0]$ generates additional vertices which are located at $t = 0$.

We can now write

$$G_{1,1}^{(\mathrm{eq})}(q;t-t') = \int_{t'}^{t} \mathrm{d}t''\, \bar{G}_{1,1}(q;t,t'')\Gamma_{1,0}^{(\mathrm{eq})}(q,t'')_{[\tilde{s}(t')]} \tag{28}$$

where $\Gamma_{1,0}^{(\mathrm{eq})}(q,t'')_{[\tilde{s}(t')]}$ is a reducible vertex function with an insertion of $\tilde{s}(t')$. Like $\bar{G}_{1,1}(q;t,t')$ it is calculated with equilibrium correlators but it contains in its last irreducible part at least one of the new 'initial vertices'.

For model A, $\Gamma_{1,0}^{(\mathrm{eq})}(q,t)_{[\tilde{s}(t')]} = \delta(t-t') + \mathrm{O}(3\text{-loop})$ (provided we set $\tau = 0$ as will be done in the following), whereas in case of model C there are already non-trivial contributions at 2-loop level. The corresponding Feynman graphs are given in figure 2.

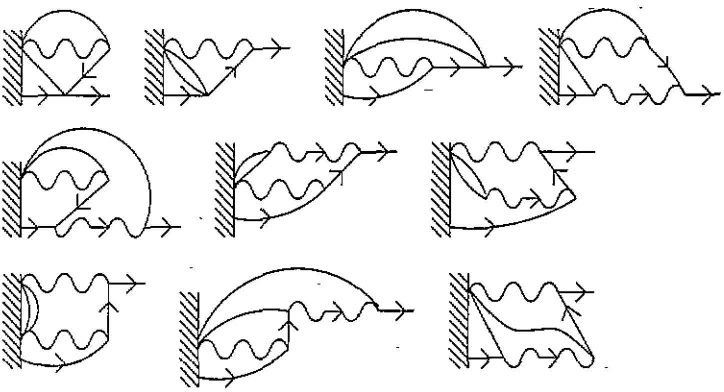

Figure 2. Feynman graphs contributing to $\Gamma_{1,0}^{(\mathrm{eq})}(q,t)_{[\tilde{s}_0]}$ at 2-loop order.

In equation (28) $\Gamma_{1,0}^{(\mathrm{eq})}(q,t)_{[\tilde{s}(t')]}$ operates as an integral kernel. This kernel can be inverted order by order in perturbation theory, i.e. we can define an inverse $K(q;t,t')$ by

$$\int_{t'}^{t} \mathrm{d}t''\, \Gamma_{1,0}^{(\mathrm{eq})}(q,t)_{[\tilde{s}(t'')]}K(q;t'',t') = \delta(t-t'). \tag{29}$$

Together with equations (27) and (28) this yields

$$G_{0,1}^{\mathrm{I}}(q,t) = \int_{0}^{t} \mathrm{d}t'\, G_{1,1}^{(\mathrm{eq})}(q;t-t')\Gamma_{1,0}(q,t')_{[\tilde{s}_0]} \tag{30}$$

where

$$\Gamma_{1,0}(q,t')_{[\tilde{s}_0]} = \int_0^{t'} \mathrm{d}t''\, K(q;t',t'')\Gamma^{(\mathrm{i})}_{1,0}(q,t'')_{[\tilde{s}_0]}. \tag{31}$$

The 2-loop result for the singular part of this 'vertex function' is given by

$$\begin{aligned}
\Gamma_{1,0}(\mathbf{0},t)_{[\tilde{s}_0]} = \delta(t) &+ \frac{\lambda(\lambda t)^{-1+\epsilon/2}}{(4\pi)^{d/2}}\left\{\frac{n+2}{12}g\left[1+\frac{\epsilon}{2}(1+\ln 2)\right]\right.\\
&- \frac{\rho\gamma^2}{2(1+\rho)}\left[1+\frac{\epsilon}{2}\left(1+\ln 2-\frac{2}{\rho-1}\ln\frac{1+\rho}{2}\right)\right]\\
&+ \left.\frac{(c_0-1)\gamma^2}{2\rho(1+\rho)}\left[1+\frac{\epsilon}{2}\left(1+2\ln(1+\rho)-\ln(2\rho)+\frac{2}{\rho-1}\ln\frac{1+\rho}{2\rho}\right)\right]\right\}\\
&+ \frac{\lambda(\lambda t)^{-1+\epsilon}}{(4\pi)^{d}}\left\{-\frac{n+2}{72\epsilon}g^2[n+14+\epsilon(n+20+(n+2)\ln 2)]\right.\\
&+ \frac{n+2}{3\epsilon}\frac{g\gamma^2}{1+\rho}\left[3+\frac{7}{2}\rho+\epsilon a_1(\rho)+(1-c_0)\left(\frac{1}{2\rho}+\epsilon a_2(\rho)\right)\right]\\
&- \frac{n}{2\epsilon}\frac{\gamma^4}{(1+\rho)^2}\left[\frac{1}{\rho}+3+4\rho+3\rho^2+\epsilon b_1(\rho)+(1-c_0)\left(\frac{1}{\rho}+2+\epsilon b_2(\rho)\right)\right]\\
&- \frac{1}{2\epsilon}\frac{\gamma^4}{(1+\rho)^2}\left[-\frac{2}{1+\rho}+7\rho^2+12\rho+6+\epsilon c_1(\rho)\right.\\
&+ \left.\left.(1-c_0)\left(\frac{2(\rho^2+\rho+1)}{\rho(1+\rho)}+\epsilon c_2(\rho)\right)+(1-c_0)^2\left(-\frac{1}{\rho^2}+\epsilon c_3(\rho)\right)\right]\right\}
\end{aligned} \tag{32}$$

a_i, b_i, and c_i are complicated functions of ρ [21] which we do not show here for general ρ since they would fill several pages of this paper. Fortunately, we only need to know their values at $\rho = 1$ in order to calculate the new exponent for a one-component order parameter ($n = 1$). In this case a_i, b_i, and c_i reduce to simple expressions:

$$\begin{aligned}
&a_1(1) = 8+5\ln 2-2\ln 3 \qquad a_2(1) = \tfrac{1}{4}+\ln 2-\tfrac{1}{2}\ln 3\\
&b_1(1) = 13+16\ln 2-7\ln 3 \qquad b_2(1) = 6\ln 2-4\ln 3\\
&c_1(1) = \tfrac{47}{2}+27\ln 2-12\ln 3 \qquad c_2(1) = 3 \qquad c_3(1) = \tfrac{1}{2}+3\ln 2-3\ln 3.
\end{aligned} \tag{33}$$

We now have to express the bare quantities on the right-hand side of equation (33) in terms of their renormalized counterparts (20). The corresponding Z factors are given in the literature [18, 17]. However, we do not yet know how to renormalize the initial correlation c_0 of the conserved field.

To obtain the missing renormalization we make use of two equations of motion. The first one,

$$0 = \int \mathcal{D}[\mathrm{i}\tilde{s}, s; \mathrm{i}\tilde{m}, m]\frac{\delta}{\delta m_0(r')}m(r,t)\exp(-\mathcal{J}[\tilde{s},s;\tilde{m},m]-\mathcal{H}^{(\mathrm{i})}[s_0,m_0])$$

gives

$$\langle m(\boldsymbol{r},t)m(\boldsymbol{r}',0)\rangle = c_0\langle m(\boldsymbol{r},t)\tilde{m}(\boldsymbol{r}',0)\rangle \qquad \text{for } t>0.$$

This relation can also be checked by an analysis of the corresponding Feynman graphs. Together with

$$\int \mathrm{d}^d r\ \langle m(\boldsymbol{r},t)\tilde{m}(\boldsymbol{r}',0)\rangle = 1$$

(which results from the conservation law) this yields

$$\int \mathrm{d}^d r\ \langle m(\boldsymbol{r},t)m(\boldsymbol{r}',0)\rangle = c_0 \qquad \text{for all } t\geqslant 0. \tag{35}$$

The second equation of motion is

$$\begin{aligned} 0 &= \int \mathcal{D}[\mathrm{i}\tilde{s},s;\mathrm{i}\tilde{m},m]\frac{\delta}{\delta\tilde{m}(\boldsymbol{r},t)}m(\boldsymbol{r}',t')\exp(-\mathcal{J}[\tilde{s},s;\tilde{m},m]-\mathcal{H}^{(\mathrm{i})}[s_0,m_0]) \\ &= \langle m(\boldsymbol{r}',t')[-\partial_t m(\boldsymbol{r},t)+\lambda\rho\Delta(\tfrac{1}{2}\gamma s^2+m-2\tilde{m})(\boldsymbol{r},t)]\rangle. \end{aligned}$$

Taking the zero-momentum limit one finds

$$\int \mathrm{d}^d r\ \partial_t\langle m(\boldsymbol{r},t)m(\boldsymbol{r}',t')\rangle = 0$$

and, by equation (35),

$$\int \mathrm{d}^d r\ \langle m(\boldsymbol{r},t)m(\boldsymbol{r}',t')\rangle = c_0 \qquad \text{for all } t,t'\geqslant 0.$$

This result implies the renormalization

$$c_0 \to \mathring{c}_0 = Z_m c_0. \tag{36}$$

We now obtain the new Z factor, Z_0, from equations (30) and (33). Since the equilibrium response function is translationally invariant the Laplace transform of $G^{\mathrm{I}}_{0,1}(q,t)$ consists of two factors. The first one is the Laplace transform of $G^{(\mathrm{eq})}_{1,1}(q,t)$ which has to be multiplicatively renormalized by $(Z_s Z_{\tilde{s}})^{-1/2}$ as usual. The second factor is the Laplace transform of $\Gamma_{1,0}(q,t)_{[\tilde{s}_0]}$. It requires an additional renormalization:

$$Z_0^{-1/2}\int_0^\infty \mathrm{d}t\ \mathrm{e}^{-\mathrm{i}\omega t}\Gamma_{1,0}(q,t)_{[\tilde{s}_0]} = \text{finite for } \epsilon\to 0.$$

We apply the minimal subtraction scheme, i.e. we absorb only the singular part of the vertex function into Z_0. In this way we get

$$\begin{aligned} Z_0 = 1 &+ \frac{n+2}{3\epsilon}U - \frac{2(1+\rho^2)}{\rho(1+\rho)}\frac{\gamma^2}{\epsilon} + c_0\frac{2\gamma^2}{\rho(1+\rho)\epsilon} \\ &+ U^2\left[\frac{(n+2)(n+5)}{9\epsilon^2} - \frac{n+2}{6\epsilon} + \frac{n+2}{3\epsilon}\ln 2\right] \\ &+ \frac{n+2}{3\epsilon}\frac{2U\gamma^2}{\rho(1+\rho)}\left[-\frac{5\rho^2+3\rho+2}{\epsilon} + A_1(\rho) + c_0\left(\frac{2}{\epsilon}+A_2(\rho)\right)\right] \\ &+ \frac{n\gamma^4}{\rho(1+\rho)^2\epsilon}\left[\frac{3\rho^3+4\rho^2+5\rho+2}{\epsilon} + B_1(\rho) + c_0\left(-\frac{1+2\rho}{\epsilon}+B_2(\rho)\right)\right] \\ &+ \frac{2\gamma^4}{\rho^2(1+\rho)^3\epsilon}\left[\frac{5\rho^5+11\rho^4+13\rho^3+6\rho^2+2\rho+1}{\epsilon} + C_1(\rho)\right. \\ &\left. + c_0\left(-\frac{4\rho^3+4\rho^2+3\rho+2}{\epsilon} + C_2(\rho)\right) + c_0^2\left(\frac{1+\rho}{\epsilon}+C_3(\rho)\right)\right] \end{aligned} \tag{37}$$

where

$$
\begin{aligned}
A_1(1) &= \tfrac{13}{4} - \ln 2 - \tfrac{5}{2}\ln 3 & A_2(1) &= -\tfrac{1}{2}(1 + \ln 2 - \ln 3) \\
B_1(1) &= -7 - 8\ln 2 + 11\ln 3 & B_2(1) &= \tfrac{3}{2} + 3\ln 2 - 4\ln 3 \\
C_1(1) &= -13 - 4\ln 2 + 15\ln 3 & C_2(1) &= \tfrac{9}{2} + 5\ln 2 - 6\ln 3 \\
C_3(1) &= -\tfrac{1}{2} - 4\ln 2 + 3\ln 3.
\end{aligned}
\tag{38}
$$

4. Scaling

With knowledge of Z_0 we are now in a position to determine the scaling behaviour of connected Green functions

$$
G^{\tilde L}_{\tilde N,N;\tilde M,M} = \langle [\tilde s_0]^{\tilde L}[\tilde s]^{\tilde N}[s]^N[\tilde m]^{\tilde M}[m]^M\rangle \tag{39}
$$

where $\tilde L$ $\tilde s$-legs are fixed to $t = 0$. This function has to be renormalized according to

$$
G^{\tilde L}_{\tilde N,N;\tilde M,M} \to \mathring{G}^{\tilde L}_{\tilde N,N;\tilde M,M} = (Z_0 Z_{\tilde s})^{\tilde L/2} Z_{\tilde s}^{\tilde N/2} Z_s^{N/2} Z_m^{-\tilde M/2} Z_m^{M/2} G^{\tilde L}_{\tilde N,N;\tilde M,M}. \tag{40}
$$

The bare Green function $\mathring{G}^{\tilde L}_{\tilde N,N;\tilde M,M}$ does not depend on the external momentum scale μ introduced in equation (20). We exploit this fact to derive the renormalization group equation (RGE) in the usual way. It reads

$$
\begin{aligned}
&[\mu\partial_\mu + \zeta\lambda\partial_\lambda + \kappa\tau\partial_\tau + \beta_U\partial_U + \beta_\gamma\partial_\gamma + \beta_\rho\partial_\rho + \tfrac{1}{2}\tilde L(\gamma_0 + \gamma_{\tilde s}) + \tfrac{1}{2}\tilde N\gamma_{\tilde s} \\
&\qquad + \tfrac{1}{2}N\gamma_s + \tfrac{1}{2}(M - \tilde M)\gamma_m - \gamma_m c_0\partial_{c_0}]G^{\tilde L}_{\tilde N,N;\tilde M,M} = 0.
\end{aligned}
\tag{41}
$$

The Wilson functions

$$
\begin{aligned}
\zeta &= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_0 \ln\lambda & \kappa &= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_0 \ln\tau & \beta_U &= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_0 U \\
\beta_\gamma &= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_0 \gamma & \beta_\rho &= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_0 \rho & \gamma_s &= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_0 \ln Z_s \\
\gamma_{\tilde s} &= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_0 \ln Z_{\tilde s} & \gamma_m &= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_0 \ln Z_m & \gamma_0 &= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_0 \ln Z_0
\end{aligned}
\tag{42}
$$

are known from the equilibrium critical dynamics except for γ_0 which follows from (38)

$$
\begin{aligned}
\gamma_0 = &-\frac{n+2}{3}U + \frac{2(1+\rho^2)}{\rho(1+\rho)}\gamma^2 - \frac{2c_0\gamma^2}{\rho(1+\rho)} \\
&+ \frac{n+2}{3}U^2(1 - 2\ln 2) - \frac{n+2}{3}\frac{4U\gamma^2}{\rho(1+\rho)}[A_1(\rho) + c_0A_2(\rho)] \\
&- \frac{2n\gamma^2}{\rho(1+\rho)^2}[B_1(\rho) + c_0B_2(\rho)] \\
&- \frac{4\gamma^4}{\rho^2(1+\rho)^3}[C_1(\rho) + c_0C_2(\rho) + c_0^2C_3(\rho)] + \mathrm{O}(3\text{-loop}).
\end{aligned}
\tag{43}
$$

At the fixed point $U_\star$, $\gamma_\star^2$, $\rho_\star$ the initial correlation c_0 scales as $c_0 \to l^{-\eta_m} c_0$ at a scale transformation $r \to lr$. This can be easily checked in the RGE. Since $\eta_m = -\tilde{\alpha}/\nu$ (where $\tilde{\alpha} = \max\{\alpha, 0\}$) and $\alpha > 0$ for a one-component order parameter in dimensions $4 > d > 2$ we may set $c_0 = c_0^\star = 0$ to obtain the asymptotic scaling behaviour

$$G^{\tilde{L}}_{\tilde{N},N;\tilde{M},M}(\{r, t; \tau; \lambda, \mu) = l^{(d-2+\eta)N/2+(d+2+\tilde{\eta})(\tilde{N}+\tilde{L})/2+\eta_0\tilde{L}/2+(d+\eta_m)M/2+(d-\eta_m)\tilde{M}/2} \times G^{\tilde{L}}_{\tilde{N},N;\tilde{M},M}(\{lr, l^z t\}; l^{-1/\nu_F}\tau; \lambda, \mu). \tag{44}$$

(ν_F is a Fisher-renormalized exponent [18].)

At the fixed point

$$\begin{aligned} u_\star &= \tfrac{1}{3}\epsilon + \tfrac{17}{81}\epsilon^2 + O(\epsilon^3) & \gamma_\star^2 &= \tfrac{1}{3}\epsilon - \tfrac{19}{81}\epsilon^2 + O(\epsilon^3) \\ U_\star &= \tfrac{4}{3}\epsilon - \tfrac{40}{81}\epsilon^2 + O(\epsilon^3) & \rho_\star &= 1 + \epsilon(-\tfrac{23}{18} + \tfrac{14}{3}\ln\tfrac{4}{3}) + O(\epsilon^2) \\ c_0^\star &= 0 \end{aligned} \tag{45}$$

equation (43) yields the new exponent (for $n = 1$)

$$\eta_0 = \gamma_0^\star = -\tfrac{2}{3}\epsilon + \left(\tfrac{73}{162} - \tfrac{46}{9}\ln 2 + \tfrac{7}{3}\ln 3\right)\epsilon^2 + O(\epsilon^3). \tag{46}$$

The short-time scaling behaviour given in the introduction follows from the short-time expansion [3]

$$\tilde{s}(t) = \tilde{\sigma}(t)\tilde{s}_0 + \cdots \tag{47}$$

$$s(t) = \sigma(t)\dot{s}_0 + \cdots. \tag{48}$$

By means of the RGE one finds at the fixed point the power-law behaviour

$$\tilde{\sigma}(t) \sim t^{-\theta} \tag{49}$$

$$\sigma(t) \sim t^{-\theta+1} \tag{50}$$

with $\theta = -\eta_0/(2z)$. This equations together with (44) tell us that the correlation and response functions satisfy the scaling laws

$$C(r; t, t') = r^{-(d-2+\eta)} \left(\frac{t}{t'}\right)^{\theta-1} F\left(\frac{r}{\xi}, \frac{t-t'}{\xi^z}, \frac{t}{t'}\right) \tag{51}$$

$$\chi(r; t, t') = r^{-(d-2+\eta+z)} \left(\frac{t}{t'}\right)^{\theta} \tilde{F}\left(\frac{r}{\xi}, \frac{t-t'}{\xi^z}, \frac{t}{t'}\right) \tag{52}$$

where the scaling functions F and $\tilde{F}$ are non-singular for $t' \to 0$.

By equations (23) and (44) we find for the autocorrelation function $\langle s(r, t)s_0(r)\rangle$ the scaling behaviour

$$C(t) = \tau_0^{-1}\langle s(r, t)\tilde{s}_0\rangle = \tau_0^{-1} t^{-(d/z-\theta')} f(t/\xi^z) \tag{53}$$

with $\theta' = \theta + (2 - z - \eta)/z$.

It is well known that the strong scaling fixed point (ρ_* finite) considered above is stable for $n = 1$ and ϵ small but it becomes unstable in other regions of the (n, d) plane. This has been discussed in detail by Brézin and De Dominicis [17].

To the right of a line

$$n = N_1(\epsilon) = 4 - \left(\tfrac{15}{4} + \tfrac{3}{2}\ln\tfrac{4}{3}\right)\epsilon + O(\epsilon^2) \tag{54}$$

the fixed point $\rho_* = \infty$ governs the critical behaviour of the system. In this limit the Wilson function γ_0 is given by

$$\gamma_0 = -\tfrac{1}{3}(n+2)u + \tfrac{1}{3}(n+2)u^2(1 - 2\ln 2) - n\gamma^2 + O(\text{3-loop}) \tag{55}$$

and we find for the critical exponent

$$\eta_0 = \eta_0^{(A)} - \frac{\tilde{\alpha}}{\nu} + O(\epsilon^3) \tag{56}$$

where $\eta_0^{(A)}$ is the value of η_0 for model A. For $n = 1$ the exponent θ as a function of dimension d is shown in figure 3.

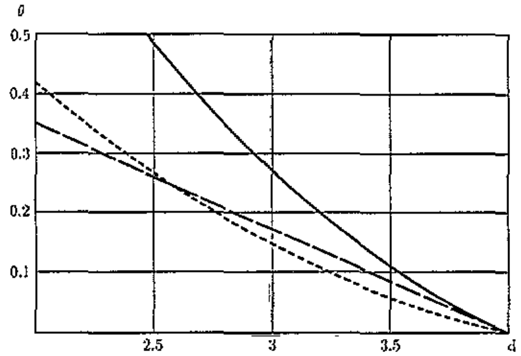

Figure 3. The exponent θ (in second order in ϵ) as a function of dimension d: full curve, model C (fixed point ρ_* finite); broken curve, model C (fixed point $\rho_* = \infty$); dotted curve, model A.

5. Conclusions and outlook

We have extended the study of the short-time scaling behaviour of critical relaxation processes to systems with a conserved quantity coupling to the order parameter.

In model C the (short-range) initial correlations of the order parameter and the conserved density have no effect on the asymptotic scaling behaviour and so it is sufficient to consider an initial state with vanishing correlations. For a one-component order parameter the fixed point of the coupling coefficient ρ (ratio of the kinetic coefficients of the conserved density and the order parameter) is finite while $\rho_* = \infty$ in a another region of the (n, d) plane. In both cases we have obtained the initial slip exponent to second order in $\epsilon = 4 - d$.

Brézin and De Dominicis [17] have shown that in a third region the fixed point $\rho_* = 0$ is attractive. After integration over the conserved field the limit $\rho \to 0$ can be performed directly in the weight $\exp(-\mathcal{J})$. This gives a dynamic functional with a new interaction which is non-local with respect to time and corresponds to the presence of quenched impurities. The relaxation of systems with this kind of disorder has been studied by Kissner [9]. However, the order of the limit $\rho \to 0$ and the ϵ expansion cannot be exchanged.

The renormalization group analysis of non-equilibrium critical relaxation presented here should be continued by a calculation of correlation and response functions and the scaling function describing the order-parameter decay.

An experimental test of our results by a quench of a real system from a high temperature to T_c is difficult since the temperature has to be stabilized sufficiently rapidly to render an observation of the initial stage of the relaxation possible. We therefore suggest performing Monte Carlo simulations to obtain 'experimental' values of the initial slip exponents which can be compared with the predictions of the ϵ expansion.

In a subsequent paper the relaxation of systems with reversible mode coupling of conserved fields to the order parameter will be studied.

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